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Radix-$R > 2$ Quantum Computation

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A quantum mechanical system is presented for which a multiple-valued quantum algebra and logic are derivable. The system is distinguished from previous quantum computational proposals by the definition of higher order quantum algebras and logics derived from multilevel quantum spin systems.

1. INTRODUCTION

The potential impact and applications of quantum computing have recently been investigated.[2] Most of the current literature considers quantum coherency to be of critical importance. This strict coherence constraint implies the necessity of isolated quantum systems which do not communicate with each other or the environment except at measurement (at which time the information content of the states is corrupted, namely the quantum measurement problem). The present work does not admit this constraint, allowing for mixed quantum states to encode and process information. The appeal of this relaxed constraint is that it may be possible to engineer a material system to perform quantum algebraic and logical operations in condensed matter systems at high temperatures, thus obviating the need to contain hundreds of ions in an ion trap at near-absolute zero temperatures.

2. A SIMPLE QUANTUM MECHANICAL FINITE-STATE MACHINE

Consider a solid system rich in nuclear and electronic spin states (e.g., transition metal-doped $Bi_{12}SiO_{20}$ as used in Hotaling (1995)). A spin Hamilto-

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[2] Several articles have appeared in the literature and on the world-wide web under the subject headings quantum computing, quantum cryptography, and quantum complexity.

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nian can be derived which allows for photonic perturbation of those electronic and nuclear spin states (Hotaling, 1996, 1997).

Conjecture 1. A finite-state machine may be realized from spin states (corresponding to logic levels) in a condensed matter medium and photonic perturbation (logical connectives) of those states.

As a simple example of a multiple-valued quantum spin system, consider a system of two particles at fixed positions in space (separated by a distance r), endowed with spin 1/2, and joined with unit vector **n.** The state of the system is given by the state vector

$$|\Psi\rangle = |S, I\rangle \tag{1}$$

where S and I correspond to the spins of the particles. The set $\{|s,i>\}$ is the basis of eigenvectors common to S_z and I_z (in the spin Hilbert space of spin states).

The magnetic moments of S and I are given by

$$M_S = \gamma_S S \tag{2}$$

and

$$M_I = \gamma_I I \tag{3}$$

Assuming S and I have differing magnetic moments, they will have different Larmor frequencies when placed in an external **B** field:

$$\omega_1 = -\gamma_1 \mathbf{B}_0 \tag{4}$$

$$\omega_2 = -\gamma_2 \mathbf{B}_0 \tag{5}$$

The unperturbed Hamiltonian for the system H'_0 is given by

$$H_0 = \omega_1 S_z + \omega_2 I_z \tag{6}$$

The interaction Hamiltonian $\mathcal{H}$ for this simplified quantum system is given by

$$\mathcal{H} = \frac{\mu_0}{4\pi} \frac{\gamma_S \gamma_I}{r^3} [S \cdot I - 3(S \cdot \mathbf{n})(I \cdot \mathbf{n})] \tag{7}$$

The state space in which $\mathcal{H}$ acts is spanned by the set $\{|\varphi_{nlm}\rangle \otimes |s,i\rangle\}$, where $|\varphi_{nlm}\rangle$ is a standard basis in the state space of one of the relative particles, and $|s,i\rangle$ is the basis of eigenvectors common to S_z and I_z.

Remark 1. The interaction Hamiltonian (dipole–dipole) of equation (7) neglects several terms, but for the purpose herein, i.e., the description of a quantum logic and algebra, equation (7) is sufficient. The reader is referred to Hotaling (1996, 1997) for a more complete spin Hamiltonian. As in Hotaling (1995), we are interested in magnetic resonance transitions of the system, since this technique allows straightforward measurement.

The total Hamiltonian is

$$\mathbb{H} = H_0 + \mathcal{H} \tag{8}$$

In equation (8) the interaction Hamiltonian term is treated as a perturbation on H_0.

Remark 2. The transitions allowed for the spin resonance condition (Hotaling, 1995) for magnetic field parallel to the lab x-axis are

$$|\uparrow, \uparrow\rangle \leftrightarrow |\downarrow, \uparrow\rangle; \qquad S_x \neq 0 \tag{9}$$

$$|\uparrow, \downarrow\rangle \leftrightarrow |\downarrow, \downarrow\rangle; \qquad S_x \neq 0 \tag{10}$$

forming a four-state quantum system with four distinct levels, depending upon which selection rule is chosen. Assume that, as experimentally observed in Hotaling (1995), incident laser radiation can induce these spin state transitions. Then, for various orientations of applied **B** field and photon energy, observable spin states are forced to change in response to incident radiation (switched). Thus, we have a finite state system with four levels. The Hilbert space of possible states containing combinations of electronic and nuclear spin state vectors is finite. Additionally, undefined states are discouraged due to strict quantum mechanical selection rules and the experimental form of the perturbation (laser radiation). At some finite energy, the atom may be ionized, which would lead to loss of data. For construction of a computational machine, the energy input to the system is strictly lower than that required to ionize the atoms of the system. This energy constraint is physically realized through the use of laser radiation as spin state perturbation.

Formally, if we assign to each of the above four quantum levels numerical values

$$|\uparrow, \uparrow\rangle \Leftrightarrow 0 \tag{11}$$

$$|\uparrow, \downarrow\rangle \Leftrightarrow 1 \tag{12}$$

$$|\downarrow, \downarrow\rangle \Leftrightarrow 2 \tag{13}$$

$$|\downarrow, \uparrow\rangle \Leftrightarrow 3 \tag{14}$$

we have the basis for a four-state (finite-state) machine logic. We must now consider a transformation from physical (measurement) space to the finite group representation thereof, the elements of which, comprise the logical states of our finite state computer: the set $\mathcal{S} = \{\beta \ni \beta \in \{0, 1, 2, 3\}\}$. Given this *ordered* set of elements, we write the following axioms:

Axiom 1. All system states are subsets of $\mathcal{S}$—Zorn's Lemma applies.

Axiom 2. Equality relation: We may write the symbol '=' as meaning that logical state β of the system physically corresponds to state $|\Psi\rangle$. Quantum mechanically, this implies that $|\Psi\rangle_i = |\Psi\rangle_j$, or two distinct atomic systems are physically in the same quantum state $|\Psi\rangle$. Further, if we assign the numerical value β to both states, we may say that an element is equal to itself,

$$\beta = \beta \tag{15}$$

Axiom 3. If two elements of S, β_i and β_j, are equal, then

$$\beta_i = \beta_j \quad \text{then} \qquad \beta_j = \beta_i \tag{16}$$

Axiom 4. There exists a transitive law:

$$\text{If} \quad \beta_i = \beta_j \quad \text{and} \quad \beta_j = \beta_k, \qquad \text{then} \quad \beta_i = \beta_k \tag{17}$$

Axiom 5. Closure. Binary operations performed upon elements of :SS yield elements of $\mathcal{S}$.

Remark 3. If are(11)–(14) are replaced by (18)–(21) below, we have a three-state logic which redefines the axioms above. This three state logic is interesting since it can be used in the context of the radix-4 quantum computer of the present work to perform Boolean logic, while the mappings of (11)–(14) are used for arithmetical operations.

$$|\uparrow, \uparrow\rangle \Leftrightarrow 1 \tag{18}$$

$$|\uparrow, \downarrow\rangle \Leftrightarrow 0^+ \tag{19}$$

$$|\downarrow, \uparrow\rangle \Leftrightarrow 0^- \tag{20}$$

$$|\downarrow, \downarrow\rangle \Leftrightarrow -1 \tag{21}$$

Axiom 6. All system states are subsets of $\mathcal{S}$—Zorn's Lemma applies.

Axiom 7. Equality relation: We can only write the symbol '=' as meaning that the logical state β of the system, which physically corresponds to $|\Psi\rangle$, is physically equal to itself IFF $\beta \in \{-1, 1\}$. Quantum mechanically, this implies that $|\Psi\rangle = |\Psi\rangle$, or is physically the same quantum state in two distinct atomic systems. However, for the zero states, 0^+ is a physically different state than 0^-. Thus we write the symbol '$\vDash$' for the three-state logical equality,

$$\beta_i \vDash \beta_j \tag{22}$$

Axiom 8. Inequality ($\nvDash$): If β_i is not equal to $\beta^{i}{}_{j}$, then we write

$$\beta_i \nvDash \beta_j \tag{23}$$

We have the choice of making the logical decision that 0^+ is logically equivalent to 0^-, or

$$0^- \models 0^+ \tag{24}$$

or

$$0^- \not\models 0^+ \tag{25}$$

If we leave this to the choosing of the programmer, then the present logic has the potential for formulation of NP-complete problms. Choosing (24) yields the three-state logic, while choosing (25) yields a four-state logic. Several three-state logics have been discussed in the literature.[3]

4. CONCLUSION

The present work has presented a proposal for a simple radix-4 finite-state quantum logic using quantum spin states as logic levels and photon-induced spin transitions as connectives of that four-state logic. A three-state logic is seen as a special case of this four-state system. The four-state logic is easily expandable to yet higher radix algebras and higher order quantum logics. This would be performed by exploiting material systems with larger numbers of measurable spin states. For example, the doped $Bi_{12}SiO_{20}$ sillenite crystals of Hotaling (1995) showed a rich ENDOR spin structure, including hyperfine and quadrupole lines. It is apparent that quantum mechanical spin systems offer the ability to perform calculations in higher radix than 2, and logic operations with more than two logical truth states. The present work thus proposes that the mathematics community consider multiple-valued algebras and quantum logics (e.g., Quantales), while concurrently, the physics community seek to exploit the vast ESR, ENDOR, and NMR spectroscopy literature for systems capable of encoding and manipulating higher radix data.

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[3] See the more than 20 years of conference proceedings of the Multi-State Logic Symposia published by the IEEE Press.